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B.Sc. (Hons) Part-1, Paper-1

Theory of Equations

The study of algebraic equations which are defined by a polynomial is called the theory of Equations.

Fundamental theorem of Algebra :-
Every polynomial equation with real coefficient has at least one root.

Theorem :- Every eqn. of n^{th} degree has n roots real or imaginary and no more

Proof :- let α_1 be the root of $f_n(x) = 0$, then $f_n(x)$ is divisible by $(x-\alpha_1)$ without remainder

$$\therefore f_n(x) = (x-\alpha_1) f_{(n-1)}(x) \quad \text{——— (I)}$$

Where quotient $f_{n-1}(x)$ is a polynomial of $(n-1)^{th}$ degree in x.

Again by fundamental theorem $f_{n-1}(x) = 0$ has a root.

let α_2 be the root of $f_{n-1}(x) = 0$. Then $f_{(n-1)}(x)$ is divisible by $(x-\alpha_2)$

$$\therefore f_{n-1}(x) = (x-\alpha_2) f_{n-2}(x)$$

Putting this value in (I), we have

$$f_n(x) = (x-\alpha_1)(x-\alpha_2) f_{n-2}(x)$$

Similarly, $f_{n-2}(x) = 0$ has a root say α_3

then $f_{n-2}(x) = (x-\alpha_3) f_{n-3}(x)$

Where $f_{n-3}(x)$ is a polynomial of $(n-3)^{th}$ degree in x

Continuing this process, we get

$f_n(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n) Q$ — (II)

Now, $f_n(x)$ and $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n)$ is of nth degree and hence Q must be independent of x

Now, $f_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$

Hence, equating the coefficient of x^n in (II)

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n) Q$$

we get $Q = a_0$

Thus from (II)

$f_n(x) = a_0 (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \cdots (x-\alpha_n)$ — (III)

Now, the R.H.S of (III) vanish when $x = \alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$

The eqn $f_n(x) = 0$ has n roots.

Now, to prove that $f_n(x) = 0$ has got n and only n roots

Let $\delta \neq \alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$ can vanish

Then no factors of $f_n(x)$ and consequently $f_n(x) \neq 0$ as is evident from (III) for $x = \delta$. Hence $f_n(x) = 0$ cannot have more than n roots

Ex. $f(x) = x^2 - 3x + 2 = 0$

$\Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x^2 - 2x - x + 2 = 0$

$\Rightarrow x(x-2) - 1(x-2) = 0 \Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$

Here $f(x)$ is equation of second degree, then we get two solution of the equation i.e. 1, 2